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Liapunov Criteria for Weak Stochastic Stability

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1. INTRODUCTION

A current problem in systems theory is to establish criteria for the stability of a dynamical system which is subjected to random perturbations. Suppose the system is described by a differential equation

$$\dot{x} = f(t, x, \xi(t)), \qquad t \geqslant 0, \tag{1}$$

where x is the state vector and $\xi(t)$ represents a random perturbation. One natural approach to the problem is to look for those stability properties of the stochastic differential equation (1) which are more-or-less closely analogous to those of the nonstochastic equation

$$\dot{x} = \bar{f}(t, x), \qquad t \geqslant 0. \tag{2}$$

where $\bar{f}(t, x) \equiv f(t, x, 0)$. For example if $f(t, 0, \xi) \equiv 0$ then (1) and (2) possess in common the null solution $x(t) \equiv 0$. Various analogoes for (1) of Liapunov stability of the null solution of (2) have been discussed in the literature [*1*], [*2*].

The system (2) is said to be Lagrange stable [*3*] if the solutions of (2) are ultimately uniformly bounded. An obvious analogue of this property is that the sample functions defined by (1) be bounded with probability 1. It turns out, however, that the requirement of boundedness excludes many stochastic models of interest, even some for which, say, all moments $\mathscr{E}\{| x(t) |^k\}$ are bounded.

Suppose the process X defined by (1) is temporally homogeneous. An alternative, 'weak' counterpart to Lagrange stability is the property that X be *recurrent* or, more strongly, that X be *positive*. Roughly speaking, X is recurrent if for every initial state, any ball in the state space is hit eventually,

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with probability 1. X is positive if, in addition, the hitting time has finite expectation. Under additional restrictions, positivity of X is equivalent to the existence of a unique invariant probability measure μ: that is, if the distribution of $x(0)$ is μ then so is that of $x(t)$, for all $t > 0$.

In this note sufficient conditions for recurrence and positivity are established for the diffusion process X defined by a stochastic differential equation of Itô's type. In addition we obtain conditions for nonrecurrence and nonpositivity. The conditions require the existence of functions which closely resemble Liapunov functions. This fact often makes it possible to infer 'weak' stability of a stochastic system by starting with a Liapunov function for Lagrange stability of a corresponding deterministic system. Using this technique we discuss a nonlinear system of Lur'e type.

2. Preliminary Results

In this section some known results are collected for ease of reference. We start with a precise version of (1), namely Itô's equation (cf. [*4*] 11, Section 2)

$$dx(t) = f(x(t))\,dt + G(x(t))\,dw(t), \qquad t \geqslant 0. \tag{3}$$

In (3), x and $f(x)$ are n-vectors, $G(x)$ is an $n \times n$ nonsingular matrix and $w(t)$ is an n-dimensional Wiener process. It is assumed that $n \geqslant 2$; the results for $n = 1$ are known [*5*] and are straightforward to apply in practice. In the following, E denotes Euclidean n-space, $|\cdot|$ the Euclidean norm, and a prime ($'$) the transpose of a vector or matrix. The following assumptions are made with respect to (3):

(a) For some constant c_1,

$$|f(x) - f(y)| + |G(x) - G(y)| < c_1 |x - y| \qquad (x, y \in E)$$

(b) $x(0)$ is a random variable independent of the process $\{w(t); t \geqslant 0\}$.

(c) For some constant $c_2 > 0$,

$$y'G(x)\,G(x)'\,y \geqslant c_2 y'y \qquad (x, y \in E).$$

Conditions (a) and (b) ensure that (3) defines an essentially unique process $X = \{x(t); t \geqslant 0\}$ with the following properties:

(A) X is a continuous Feller process ([*4*] Theorem 11.4): that is, (i) X is a strong Markov process ([*4*] Theorem 3.10); (ii) the sample functions $x(t)$, $t \geqslant 0$, are almost all continuous; (iii) if $u(x)$ is a bounded continuous function of $x \in E$ then so is $T_t u(x) = \int_E P(t, x, dy)\, u(y)$ for every $t \geqslant 0$. Here $P(t, x, \Gamma)$ is the transition function of X.

(B) If $u(x)$ is a twice-continuously differentiable function of $x \in E$ with compact support, then the limit

$$\lim_{t \downarrow 0} t^{-1} \left[\int_E P(t, x, dy)\, u(y) - u(x) \right]$$

exists. Its value is

$$\mathscr{L}[u(x)] = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} f_i(x) \frac{\partial u(x)}{\partial x_i},$$

where a_{ij} is the (i, j)th element of GG' ([*4*] Theorem 11.5).

The operator $\mathscr{L}$ is the *differential generator* of X. Condition (c) implies that $\mathscr{L}$ is a strictly elliptic operator for $x \in E$. Intuitively this condition means that each component of the state vector x is directly influenced by the random increments dw; in other words the system is, in a local sense, 'controllable with respect to the white noise $\dot{w}$.' In the absence of a less stringent (yet simple) criterion for controllability of a nonlinear system, it is convenient to impose (c) in order to obtain the stronger property:

(A′) X is a continuous, strongly Feller process. That is, X has properties (i) and (ii) of (A) together with (iii)′: if $u(x)$ is a bounded measurable function of $x \in E$ then $T_t u(x)$ is a bounded continuous function of $x \in E$ for each $t > 0$. This fact is proved in the Appendix.

The stability criteria of Sections 3 and 4 are based on certain results of Khas'minskii [*6*]. It can be verified (see Appendix) that if conditions (a)-(c) hold, then the assumptions made in [*6*] are valid. From now on we assume that (a)-(c) are satisfied.

Let $P_x(\mathscr{E}_x)$ denote probability measure (expectation) on the probability space of X when $x(0) = x = \text{const.}$ with probability 1. The process X is *recurrent* if there exists a compact subset $K \subset E$ such that, for every $x \in E$,

$$P_x\{x(t) \in K \text{ for some } t\} = 1.$$

Let G be a nonempty open set in E and let τ_G be the first time the boundary of G is reached. X is *positive* if it is recurrent and if $\mathscr{E}_x \tau_G < \infty$, for arbitrary $G \subset E$ and $x \in E - G$.

The boundary Γ of a compact set in E is *smooth* if it is representable locally in the form

$$x_n = g(x_1, \cdots, x_{n-1}),$$

where g is a function of Hölder class $C_{2+\alpha}$ (for the detailed definition see [*7*] IV, Section 7). A *normal domain* is a nonempty, open, bounded and simply connected set in E with smooth boundary. A function $u = u(x)$ defined on the closure $\bar{G}$ of an open set $G \subset E$ is *smooth* if it is of Hölder class $C_{2+\alpha}$ on the compact subsets of $\bar{G}$.

The following basic results are due to Khas'minskii [6].

LEMMA 1 [6]. *Let G be a normal domain with boundary Γ. X is recurrent if and only if the equation*

$$\mathscr{L}[u(x)] = 0, \qquad x \in E - \bar{G}$$

has a unique smooth bounded solution in $E - G$ for arbitrary smooth boundary values on Γ.

LEMMA 2 [6]. *The process X is positive if and only if* (i) *X is recurrent and* (ii) *there is a normal domain G such that the equation*

$$\mathscr{L}[u(x)] = -1, \qquad x \in E - \bar{G} \tag{4}$$

has a smooth positive solution in $E - G$.

It is shown in [6] that if X is positive then there exists a unique *invariant* probability measure μ on the Borel sets of E: namely, for every Borel set $B \subset E$,

$$\mu(B) = \int_E P(t, x, B)\, \mu(dx).$$

Suppose in addition that the coefficients $f(x)$ and $G(x)\, G(x)'$ are, respectively, once and twice continuously differentiable. Then there exists a probability density $p(x)$ defined for $x \in E$, such that

$$\mu(B) = \int_B p(x)\, dx_1 \cdots dx_n .$$

Moreover $p(x)$ is the normalized positive solution of the stationary Fokker-Planck equation

$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial_2\{[G(x)\, G(x)']_{ij}\, p(x)\}}{\partial x_i\, \partial x_j} - \sum_{i=1}^{n} \frac{\partial\{f_i(x)\, p(x)\}}{\partial x_i} = 0.$$

3. Stability Criteria

We first introduce a class of real-valued functions $V(x)$, analogous to Lyapunov functions, with the following properties:

P_1: V is defined for $x \in \bar{D}_v$ where $D_v = \{x : | x | > R\}$ ($0 < R < \infty$ is arbitrary).

P_2: V is continuous in $\bar{D}_v$ and is twice continuously differentiable in D_v.

P_3: $V(x) \geqslant 0$, $x \in \bar{D}_v$ and $V(x) \to +\infty$ as $| x | \to \infty$.

THEOREM 1. *If there exists a function V with properties $P_1 - P_3$ and if*

$$\mathscr{L}[V(x)] \leqslant 0, \qquad x \in D_v,$$

then the process X is recurrent.

Proof. It will be verified that the condition of Lemma 1 holds. Let G be a normal domain with boundary Γ; we can assume that $| x | < R$ if $x \in \Gamma$.

1. Let u be a smooth function such that

$$\mathscr{L}[u(x)] = 0, \qquad x \in E - \bar{G}$$
$$u(x) = 0, \qquad x \in \Gamma$$
$$u \text{ is bounded in } E - G.$$

We will show that $u(x) \equiv 0$, $x \in E - G$. Suppose on the contrary that $u(x_0) \neq 0$ for some $x_0 \in E - G$. We can assume that $| x_0 | = R$: otherwise let $u(x) = 0$, $| x | = R$, and suppose $u(x_0) < 0$ for some

$$x_0 \in D_0 = (E - \bar{G}) \cap \{x : | x | < R\}.$$

By the strong minimum principle for harmonic functions ([7] IV, Section 4) $u(x) \equiv u(x_0)$ if $x \in \bar{D}_0$, a contradiction. If $u(x_0) > 0$ the same argument applies to $-u(x)$.

Now put $M = \max\{u(x) : | x | = R\}$ and suppose $M > 0$. By the maximum principle ([7] IV) $u(x) \leqslant M$, $x \in \bar{D}_0$. Let $\epsilon > 0$ and put

$$W(x) = \epsilon V(x) - u(x) + M, \qquad x \in D_v.$$

Then $W(x) \geqslant 0$, $| x | = R$ and $\mathscr{L}[W(x)] \leqslant 0$, $| x | > R$. Since u is bounded, $W(x) \to \infty$: choose $R' > R$ such that $W(x) > 0$ if $| x | \geqslant R'$. By the minimum principle for superharmonic functions ([7] IV), $W(x) \geqslant 0$ if $R \leqslant | x | \leqslant R'$ and so $W(x) \geqslant 0$ if $| x | \geqslant R$. Therefore $u(x) \leqslant M + \epsilon V(x)$ if $| x | \geqslant R$; since $\epsilon > 0$ is arbitrary, $u(x) \leqslant M$, $| x | \geqslant R$. Hence u attains its maximum in the interior of every domain $(E - G) \cap \{x : | x | \leqslant R''\}$ where $R'' > R$ is arbitrary. By the strong maximum principle $u(x) = \text{const.} = 0$ in every such domain and thus $u(x) \equiv 0$, $x \in E - G$.

If $M \leqslant 0$ then $m = \min\{u(x) : | x | = R\} < 0$, and the foregoing argument applies to the function $-u(x)$.

2. Consider the exterior Dirichlet problem

$$\mathscr{L}[u(x)] = 0, \qquad x \in E - \bar{G}$$
$$u(x) = \phi(x), \qquad x \in \Gamma, \tag{5}$$

where u is assumed to be smooth in $E - G$ and ϕ is smooth on Γ. We shall construct a bounded solution of (5); by the first part of the proof this solution is unique.

Let

$$m = \min\{\phi(x) : x \in \Gamma\}, \qquad M = \max\{\phi(x) : x \in \Gamma\},$$

and put

$$D_n = (E - \bar{G}) \cap \{x : |x| < R + n\}, \qquad n = 1, 2, \cdots .$$

The domain D_n is connected and has a smooth boundary. By Schauder's theorem ([7] IV) there exists a unique smooth function $u_n = u_n(x)$ defined for $x \in \bar{D}_n$ such that

$$\mathscr{L}[u_n(x)] = 0, \qquad x \in D_n$$
$$u_n(x) = \phi(x), \qquad x \in \Gamma$$
$$u_n(x) = m, \qquad |x| = R + n.$$

By the maximum principle $m \leqslant u_n(x) \leqslant M$, $x \in \bar{D}_n$. Let

$$v_n(x) = u_{n+1}(x) - u_n(x),$$

$x \in \bar{D}_n$. Then $\mathscr{L}[v_n(x)] = 0$, $x \in D_n$; $v_n(x) = 0$, $x \in \Gamma$; and $v_n(x) \geqslant 0$, $|x| = R + n$. Again by the maximum principle, $v_n(x) \geqslant 0$, $x \in \bar{D}_n$. Let N be fixed. The sequence $\{u_n(x) : n \geqslant N, x \in \bar{D}_N\}$ is nondecreasing and uniformly bounded above. By the compactness principle for harmonic functions ([7] IV, Section 4) $u(x) = \lim u_n(x)$ exists and is a smooth solution of (5) for $x \in \bar{D}_N$. Since u is obviously independent of N, u is well-defined for all $x \in E - G$, and the conclusion follows.

THEOREM 2. *If there exists a function V with properties $P_1 - P_3$ and if*

$$\mathscr{L}[V(x)] \leqslant -1, \qquad x \in D_v,$$

then the process X is positive.

Proof. It will be verified that the conditions of Lemma 2 hold. By Theorem 1, X is recurrent, and it is enough to construct a smooth positive solution of (4). Let $G = \{x : |x| < R\}$ and define $D_n = \{x : R < |x| < R + n\}$, $n = 1, 2, \cdots$. Applying Schauder's theorem as in the proof of Theorem 1 we obtain a sequence of smooth functions u_n such that $\mathscr{L}[u_n(x)] = -1$,

$x \in D_n$; $u_n(x) = 0$, $x \in \Gamma$; and $u_n(x) = 0$, $|x| = R + n$. By the minimum principle $0 \leqslant u_n(x) \leqslant u_{n+1}(x) \leqslant V(x)$, $x \in \bar{D}_n$. Reasoning as in the proof of Theorem 1, we apply the compactness principle to conclude that $u(x) = \lim u_n(x)$ is a smooth solution of (4) and $u(x) \geqslant 0$, $x \in E - G$.

4. Instability Criteria

In this section sufficient conditions are given for the process X to be nonrecurrent or at least nonpositive. We first introduce functions V with properties P_1, P_2 of Section 3, and the additional properties

P_4: V is bounded above for $x \in D_v$.

P_5: There is a normal domain G with boundary Γ such that $D_v \supset E - G$ and $\max\{V(x) : x \in \Gamma\} < \sup\{V(x) : x \in E - G\}$

P_6: $\mathscr{L}[V(x)] \geqslant 0$, $x \in D_v$.

Theorem 3. *If there exists a function V with properties P_1, P_2, $P_4 - P_6$, then the process X is nonrecurrent.*

Proof. We observe that the second part of the proof of Theorem 1 remains valid under the present hypotheses. Hence by Lemma 1 it is sufficient to show that the problem

$$\begin{aligned} \mathscr{L}[u(x)] &= 0, \qquad x \in E - \bar{G} \\ u(x) &= 0, \qquad x \in \Gamma \end{aligned} \tag{6}$$

has a smooth nontrivial solution which is bounded in $E - G$.

Let

$$M_1 = \max\{V(x) : x \in \Gamma\} \qquad \text{and} \qquad M_2 = \sup\{V(x) : x \in E - G\}.$$

Define

$$\tilde{V}(x) = (M_2 - M_1)^{-1}[V(x) - M_1], \qquad x \in E - G.$$

By P_5, $\tilde{V}(x) \leqslant 0$, $x \in \Gamma$; $\tilde{V}(x) > 0$ for some $x \in E - G$; and $\tilde{V}(x) < 1$ for all $x \in E - G$. Choose R' such that $|x| < R'$ if $x \in \Gamma$ and let

$$D_n = (E - G) \cap \{x : |x| < R' + n\}, \qquad n = 1, 2, \cdots.$$

By Schauder's theorem there exists a sequence of smooth functions u_n such that $\mathscr{L}[u_n(x)] = 0$, $x \in D_n$; $u_n(x) = 0$, $x \in \Gamma$; and $u_n(x) = 1$, $|x| = R' + n$. By the maximum principle.

$$0 \leqslant u_{n+1}(x) \leqslant u_n(x) \leqslant 1, \qquad x \in \bar{D}_n.$$

Since

$$\mathscr{L}[u_n(x) - \tilde{V}(x)] \leqslant 0, \qquad x \in D_n,$$

it follows, again by the maximum principle, that $u_n(x) \geqslant \tilde{V}(x)$, $x \in \bar{D}_n$. For every fixed N the sequence $\{u_n, n \geqslant N\}$ is nonincreasing and $u_n(x) \geqslant \tilde{V}(x)$, $x \in \bar{D}_N$. Hence, by the compactness principle, $u(x) = \lim u_n(x)$ exists and is a smooth solution of (6) for $x \in \bar{D}_N$. Clearly u is independent of N and thus is well-defined for all $x \in E - G$. Since $u(x) \geqslant \tilde{V}(x)$, and $\tilde{V}(x) > 0$ for some x, the conclusion follows.

The following theorem is sometimes useful to identify processes which are recurrent but not positive. Let $V_1(x)$, $V_2(x)$ be a pair of functions with properties P_1, P_2 in a domain $D_v = \{x : |x| > R\}$, and with the additional properties:

P_7: There is a sequence $\{x_n\}$ in D_v such that $|x_n| \to \infty$ and $V_1(x_n) \to +\infty$.

P_8: $V_2(x) > 0$, $x \in D_v$.

P_9: $\overline{\lim\limits_{\rho\to\infty}} \dfrac{\max\{V_1(x) : |x| = \rho\}}{\min\{V_2(x) : |x| = \rho\}} = 0$

P_{10}: $\mathscr{L}[V_1(x)] \geqslant 0$, $\mathscr{L}[V_2(x)] \leqslant +1$, $x \in D_v$.

THEOREM 4. *If there exists a pair of functions V_1, V_2 with properties P_1, P_2, P_7-P_{10}, then the process X is not positive.*

Proof. We assume that X is recurrent, otherwise the conclusion is obvious. It will be shown that the second condition of Lemma 2 is violated. Let G be a normal domain with boundary Γ, and let u be a smooth function on $E - G$ such that

$$\mathscr{L}[u(x)] = -1, \qquad x \in E - \bar{G}$$

$$u(x) \geqslant 0, \qquad x \in E - G$$

$$u(x) = \phi(x) \geqslant 0, \qquad x \in \Gamma.$$

The following assumptions can be made without loss of generality: $E - G \subset D_v$; the sequences $|x_n|$, $V_1(x_n)$ are strictly increasing; $|x| < |x_1|$ if $x \in \Gamma$; and $V_1(x) \leqslant 0$ if $x \in \Gamma$. Define

$$D_n = (E - \bar{G}) \cap \{x : |x| < |x_n|\}$$

and

$$\Gamma_n = \{x : |x| = x_n\},$$

$(n = 1, 2, \cdots)$. Let $u_n(x)$, $x \in D_n$, be the unique smooth solution of

$$\begin{aligned} \mathscr{L}[u_n(x)] &= -1, & x &\in D_n \\ u_n(x) &= \phi(x), & x &\in \Gamma \\ u_n(x) &= 0, & x &\in \Gamma_n . \end{aligned}$$

By the minimum principle, $0 \leqslant u_n(x) \leqslant u_{n+1}(x) \leqslant u(x)$, $x \in D_n$, and therefore

$$u_\nu(x) \leqslant u(x), \qquad x \in D_n, \qquad \nu \geqslant n. \tag{7}$$

Put $M_n = \max\{V_1(x) : x \in \Gamma_n\}$, $n = 1, 2, \cdots$. By P_7, $M_n \to \infty$ as $n \to \infty$ and, by choosing a subsequence if necessary, we can arrange that $M_1 > 0$ and $M_{n+1} > M_n$ $(n = 1, 2, \cdots)$. Put

$$V(x, \lambda) = \lambda V_1(x) - V_2(x), \qquad x \in E - G,$$

and define

$$\lambda_n = M_n^{-1} \min\{V_2(x) : x \in \Gamma_n\}.$$

Since $\lambda_n > 0$ we have $V(x, \lambda_n) \leqslant 0 \leqslant u_n(x)$, $x \in \Gamma \cup \Gamma_n$. Also

$$\mathscr{L}[u_n(x) - V(x, \lambda_n)] \leqslant 0, \qquad x \in D_n,$$

so that, by the minimum principle,

$$u_n(x) - V(x, \lambda_n) \geqslant 0, \qquad x \in D_n . \tag{8}$$

P_9 implies that for every $\lambda > 0$ there exists $N = N(\lambda)$ such that

$$\lambda M_n - \min\{V_2(x) : x \in \Gamma_n\} \leqslant 0$$

for all $n > N$. Therefore $\lambda_n \geqslant \lambda$ $(n > N)$. It follows that $\lambda_n \to \infty$ as $n \to \infty$; hence $V(x_1, \lambda_n) \to +\infty$ $(n \to \infty)$ and therefore, by (8),

$$u_n(x_1) \to +\infty \qquad (n \to \infty). \tag{9}$$

Thus (9) furnishes a contradiction to (7), and the proof is complete.

5. Application

Let X be defined by

$$\begin{aligned} dx &= Fx\,dt - b\phi(\sigma)\,dt + G(x)\,dw \\ \sigma &= c'x \end{aligned} \tag{10}$$

In (10) F is a constant $n \times n$ matrix, b and c are constant n-vectors, and ϕ is a scalar-valued nonlinear function of its scalar argument.

The nonstochastic differential equation

$$\dot{x} = Fx - b\phi(\sigma)$$
$$\sigma = c'x \tag{11}$$

has been studied extensively (see e.g. [8]). The *Lur'e problem* is to give conditions on the parameters of (11) which guarantee that the null solution of (11) is asymptotically stable in the large. For the stochastic system (10) we shall adapt Popov's solution [8] of the Lur'e problem to obtain a sufficient condition that the process X be positive.

THEOREM 5. *Let the system defined by* (10) *satisfy the following conditions:*

(i) *All the eigenvalues of F have negative real parts.*

(ii) $\sigma\phi(\sigma) > 0$ *if* $|\sigma|$ *is sufficiently large;* $\phi(\sigma)$ *is continuously differentiable; and* $|d\phi(\sigma)/d\sigma|$ *is bounded,* $-\infty < \sigma < \infty$.

(iii) *There exist two nonnegative constants* α *and* β *such that*

$$\alpha + \beta > 0$$

and

$$\operatorname{Re}(\alpha + i\omega\beta)\, c'(i\omega I - F)^{-1} b > 0$$

for all real ω.

(iv) $G(x)$ *satisfies the conditions of Section 2 and, in addition,* $|G(x)|$ *is bounded for* $x \in E$.

Then the process X is positive.

Proof. To satisfy the conditions of Theorem 2 we assume that V is of the form

$$V(x) = x'Px + \beta \int_0^{c'x} \phi(\sigma)\, d\sigma.$$

By a result of Meyer [9], there exist positive definite matrices P and Q such that

$$[Fx - b\phi(c'x)]' \frac{\partial V(x)}{\partial x} \leqslant -x'Qx, \qquad x \in E. \tag{12}$$

Furthermore

$$\frac{1}{2} \sum_{ij} [G(x)\, G'(x)]_{ij} \frac{\partial^2 V(x)}{\partial x_i\, \partial x_j} = tr[G(x)\, G(x)'\, P] + \frac{1}{2} \beta\, |G(x)'\, c|^2 \frac{d\phi(c'x)}{d\sigma}. \tag{13}$$

Since the right side of (13) is bounded it follows on adding (12) and (13) that $\mathscr{L}[V(x)] \leqslant -1$ for all sufficiently large $|x|$. This completes the proof.

Appendix

We will check that the assumptions made in Section 1 of [*6*] are valid for the process X if conditions (a)-(c) of the present note (Section 2) are satisfied. On referring to [*6*] and to the references given in Section 2, it will be seen that it is sufficient to check assumptions 1°-3° of [*6*]. Assumption 1° follows by [*4*]: Theorem 11.5, Theorem 11.4 (part 6), and Theorem 3.9′. Assumption 2° is the strong Feller property, which we now verify.

Proposition. *If conditions* (a)-(c) *of Section 2 hold, then the process X is strongly Feller.*

Proof. Let $P(t, x, B)$ be the transition function of X. We first show that $P(t, x, B)$ is continuous in x for fixed $t > 0$ and B a bounded Borel set. For $K < \infty$ let $\hat{X}$ be the process defined as in Section 2, with coefficients $\hat{f}(x)$, $\hat{G}(x)$, where

$$\hat{f}(x) = \begin{cases} f(x), & |f(x)| \leqslant K \\ K \dfrac{f(x)}{|f(x)|}, & |f(x)| \geqslant K \end{cases}$$

with a similar definition for $\hat{G}(x)$. Let $R < \infty$ be fixed and write $S = \{x : |x| < R\}$. We choose $K < \infty$ so that $\hat{f}(x) = f(x)$, $\hat{G}(x) = G(x)$ if $x \in \bar{S}$. By [*4*], Theorems 11.4 and 11.6, the process $\hat{X}$ is strongly Feller and coincides with X in the set $\bar{S}$. Let $x, y \in S$, $B \subset S$, be fixed. In obvious notation,

$$\begin{aligned} |P(t, x, B) - P(t, y, B)| \leqslant\ & |\hat{P}(t, x, B) - \hat{P}(t, y, B)| \\ & + |\hat{P}(t, x, B) - P(t, x, B)| \\ & + |\hat{P}(t, y, B) - P(t, y, B)|. \end{aligned}$$

Since $\hat{P}(t, \cdot, B)$ is continuous, it is enough to show that

$$\phi(x) = |\hat{P}(t, x, B) - P(t, x, B)|$$

can be made arbitrarily small by taking R sufficiently large. To show this assume $x(0) = \hat{x}(0) = x$ and let A be the event

$$A = \{x(\tau) \in S, 0 \leqslant \tau \leqslant t\}.$$

With P_x defined as in Section 2, we have in obvious notation

$$P(t, x, B) = P_x(x_t \in B, A) + P_x(x_t \in B, -A)$$
$$= \hat{P}_x(\hat{x}_t \in B, \hat{A}) + P_x(x_t \in B, -A).$$

Therefore

$$\begin{aligned}\phi(x) &\leqslant \hat{P}_x(\hat{x}_t \in B, -\hat{A}) + P_x(x_t \in B, -A)\\ &\leqslant \hat{P}_x(-\hat{A}) + P_x(-A)\\ &= 2P_x(-A)\\ &= 2P_x[\max_{0 \leqslant \tau \leqslant t} | x(\tau) | \geqslant R].\end{aligned}$$

By condition (a), Section 2, $| f(x) | + | G(x) | < \text{const.}\ (1 + | x |)$, $x \in E$. It follows just as in ([10] VI, Section 3) that

$$\mathscr{E}_x\{\max_{0 \leqslant \tau \leqslant t} | x(\tau) |^2\}$$

is finite, and in fact, bounded on compact subsets of E. By Chebyshev's inequality, $\phi(x) = 0(R^{-2})$ $(R \to \infty)$.

We have shown that $P(t, x, B)$ is continuous. Now suppose $f(x)$ is measurable and $| f(x) | < K < \infty$, $x \in E$. Let $f_n(x)$, $x \in E$, be a sequence of simple functions such that $f_n(x) = 0$, $| x | > n$, and

$$\sup\{| f(x) - f_n(x) | : | x | \leqslant n\} \to 0 \qquad (n \to \infty). \tag{14}$$

Let x be fixed and suppose $| x' - x | < 1$.

We have

$$\left| \int_E [P(t, x, dy) - P(t, x', dy)] f(y) \right|$$

$$\begin{aligned}&\leqslant \left| \int_{|y|>n} [P(t, x, dy) - P(t, x', dy)] f(y) \right|\\ &+ \left| \int_{|y|\leqslant n} [P(t, x, dy) - P(t, x', dy)] [f(y) - f_n(y)] \right|\\ &+ \left| \int_{|y|\leqslant n} [P(t, x, dy) - P(t, x', dy)] f_n(y) \right|\\ &= J_1 + J_2 + J_3 .\end{aligned}$$

By a previous estimate, $P(t, x', \{y : | y | > n\}) = 0(n^{-2})$ uniformly for $| x' - x | < 1$, and therefore $J_1 < \epsilon$ if $n > N_1$. By (14) $J_2 < \epsilon$ if $n > N_2$.

With $n > \max(N_1, N_2)$ fixed we have, since the f_n are simple, $J_3 < \epsilon$ if $|x - x'| < \delta$. This completes the proof.

It remains to check assumption 3° of [6]: namely $P(t, x, U) > 0$ for every $t > 0$, $x \in E$ and nonempty open $U \subset E$. In the notation of the preceding proof, let $x \in S$, $U \subset S$. Then

$$\begin{aligned} P(t, x, U) &\geqslant P_x(x_t \in U, A) \\ &= \hat{P}_x(\hat{x}_t \in U, A). \end{aligned}$$

According to [4], Theorem 13.18, the last probability is given by

$$\int_U \tilde{p}(t, x, y)\, dy,$$

where $\tilde{p}$ is the transition density corresponding to the restriction of the $\hat{X}$ process to $\bar{S}$. By property (c), Section 2, and the strong maximum principle for parabolic operators, $\tilde{p} > 0$ for $t > 0$ and $x, y \in S$, and the result follows.

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